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DYNAMICAL INVARIANT FOR FORCED TIME-DEPENDENT
HARMONIC OSCILLATOR

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Abstract

A   dynamical   invariant   is   presented   by   the   algebraic
derivation. The physical meaning of the   obtaining   invariant   is
given for the real system: the forced betatron oscillation seen in
accelerators and storage rings.

1. Introduction

It has been first obtained by Courant and Snyder[1] and strictly proved by several authors[2],[3],[4],[5],[6] that a conserved quantity for the time-dependent harmonic oscillator is given by

$$I = \frac{1}{2\beta(s)} \left[x^2 + \left(\frac{\dot{\beta}(s)}{2} x - \beta(s)\dot{x} \right)^2 \right], \qquad (1)$$

where x satisfies the equation

$$\ddot{x} + K(s)\cdot x = 0, \qquad (2)$$

and $\beta(s)$ is the arbitrary solution of the auxiliary equation

$$\frac{1}{2}\beta\ddot{\beta} - \frac{1}{4}\dot{\beta}^2 + K(s)\beta^2 = 1 \qquad (3)$$

The conserved quantity for the forced time-dependent harmonic oscillator

$$\ddot{x} + K(s)\cdot x = f(s),$$

where f(s) is the external force dependent on s alone, has been obtained in the form of the "affin" invariant, for a more general case including the damping term[7]. But it seems to be difficult to obtain any useful informations of motion from a "affin" invariant, since the "affin" invariant for the most simple system(2) is the well-known identity

$$x_1(s)\cdot\dot{x}_2(s) - \dot{x}_1(s)\cdot x_2(s) = \text{const}, \qquad (4)$$

where $x_1(s)$, $x_2(s)$ are arbitrary solutions of Eq.(2)[8]; the

identity (4) only means the fact that Eq.(2) is the linear differential equation for x.

So in the present paper we demonstrate that such a system also has a dynamical invariant of quadratic type under the help of a new auxiliary equation, by a straightforward application of dynamical algebra/5/. As an example of the present system, we choose the betatron oscillation which is the typical motion of a charged particle in accelerators and storage rings. We make clear the physical meaning of the dynamical invariant and note the relation between the solutions of two auxiliary equations and the betatron amplitude function, the equilibrium orbit, which are well known in accelerator physics/9/.

2. Derivation of Invariant

We can construct easily the dynamical algebra for the Hamiltonian

$$H(x,p;s) = \sum_{n=1}^{N} h_n(s) \cdot \Gamma_n(x,p), \tag{5}$$

following the usual procedure. Here the dynamical algebra is the Lie algebra of phase space function Γ_n, which are closed under the action of the Poisson bracket [,]:

$$[\Gamma_n, \Gamma_m] = \sum_{r=1}^{N} C_{n.m}^{r} \Gamma_r \tag{6}$$

where the $C_{n.m}^{r}$ are the structure constants of the algebra. For the Hamiltonian(5), the dynamical algebra consists of $\Gamma_1 = 1$, $\Gamma_2 = p$, $\Gamma_3 = x$, $\Gamma_4 = \frac{1}{2}p^2$, $\Gamma_5 = px$, and $\Gamma_6 = \frac{1}{2}x^2$, with the Poisson bracket

$$[\; \Gamma_1 \, , \, \Gamma_i \;] \;=\; 0 \qquad i=1,\cdots,6 \;, \qquad [\; \Gamma_3 \, , \, \Gamma_5 \;] \;=\; \Gamma_3 \;,$$

$$[\; \Gamma_2 \, , \, \Gamma_3 \;] \;=\; -\Gamma_1 \;, \qquad\qquad\qquad [\; \Gamma_3 \, , \, \Gamma_6 \;] \;=\; 0 \;,$$

$$[\; \Gamma_2 \, , \, \Gamma_4 \;] \;=\; 0 \;, \qquad\qquad\qquad\quad [\; \Gamma_4 \, , \, \Gamma_5 \;] \;=\; -\Gamma_4 \;, \qquad (7)$$

$$[\; \Gamma_2 \, , \, \Gamma_5 \;] \;=\; -\Gamma_2 \;, \qquad\qquad\qquad [\; \Gamma_5 \, , \, \Gamma_6 \;] \;=\; -2\Gamma_6 \;,$$

$$[\; \Gamma_2 \, , \, \Gamma_6 \;] \;=\; -\Gamma_3 \;, \qquad\qquad\qquad [\; \Gamma_6 \, , \, \Gamma_4 \;] \;=\; \Gamma_2 \;,$$

$$[\; \Gamma_3 \, , \, \Gamma_4 \;] \;=\; \Gamma_2 \;.$$

The structure constants $C^r_{n.m}$ may be described by the matrices

$$C^1_{n.m}=\begin{pmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{matrix} & \Large 0 \\[4pt] \Large 0 & \Large 0 \end{pmatrix} \quad C^2_{n.m}=\begin{pmatrix} \Large 0 & \begin{matrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{matrix} \\[4pt] \begin{matrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} & \Large 0 \end{pmatrix} \quad C^3_{n.m}=\begin{pmatrix} \Large 0 & \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{matrix} \\[4pt] \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{matrix} & \Large 0 \end{pmatrix}$$

$$C^4_{n.m}=\begin{pmatrix} \Large 0 & \Large 0 \\[4pt] \Large 0 & \begin{matrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{pmatrix} \quad C^5_{n.m}=\begin{pmatrix} \Large 0 & \Large 0 \\[4pt] \Large 0 & \begin{matrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \end{pmatrix} \quad C^6_{n.m}=\begin{pmatrix} \Large 0 & \Large 0 \\[4pt] \Large 0 & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{matrix} \end{pmatrix}$$

(8)

The time development of a phase-space function I is given by

$$\frac{dI}{ds} \;=\; \frac{\partial I}{\partial s} \;+\; [\; I \, , \, H \;] \tag{9}$$

and the dynamical invariant I is characterized by

$$\frac{dI}{ds} \;=\; 0 \;, \qquad \text{or} \qquad \frac{\partial I}{\partial s} \;+\; [\; I \, , \, H \;] \;=\; 0 \tag{10}$$

We now look for an invariant, which is a member of the dynamical algebra

$$I \;=\; \sum_{n=1}^{6} \; \lambda_n(s)\, \Gamma_n \tag{11}$$

which gives

$$\sum_{r=1}^{6} \left[\dot{\lambda}_r + \sum_{n,m}^{6} C^{r}_{n,m} h_m(s) \lambda_n \right] \Gamma_r = 0 , \tag{12}$$

with $\qquad h_1(s) = 0 \qquad , \qquad h_4(s) = 1 \qquad ,$

$\qquad\qquad h_2(s) = 0 \qquad , \qquad h_5(s) = 0 \qquad ,$

$\qquad\qquad h_3(s) = -f(s) , \text{and} \qquad h_6(s) = K(s) \qquad .$

and therefore the system of linear first-order equations

$$\dot{\lambda}_r + \sum_{n=1}^{6} \left(\sum_{m=1}^{6} C^{r}_{nm} h_m(s) \right) \lambda_n = 0 \tag{13}$$

The coefficients $\lambda_n(s)$ of the dynamical invariant

$$I = \lambda_1(s) + \lambda_2(s)p + \lambda_3(s)x + \frac{\lambda_4(s)}{2} p^2 + \lambda_5(s)px + \frac{\lambda_6(s)}{2} x^2, \tag{14}$$

are solution of the differential equations

$$\frac{d}{ds} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} = \begin{pmatrix} 0 & A(s) & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & A(s) & 0 & 0 \\ 0 & K(s) & 0 & 0 & A(s) & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & K(s) & 0 & -1 \\ 0 & 0 & 0 & 0 & 2K(s) & 0 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} \tag{15}$$

where $A(s)$ is $h_3(s)$.

Setting $\lambda_4 = \beta_c(s)$, we find

$$\lambda_5 = -\frac{1}{2} \dot{\beta}_c \tag{16}$$

$$\dot{\lambda}_6 = -K(s)\dot{\beta}_c \tag{17}$$

$$\lambda_6 = \frac{1}{2} \ddot{\beta}_c + K(s)\beta_c \tag{18}$$

Equating the s derivative of (18) with (17), we finally obtain

$$\dddot{\beta}_c + 4K(s)\dot{\beta}_c + 2\dot{K}(s)\beta_c = 0 \tag{19}$$

which has the integral

$$\frac{1}{2}\beta_c\,\ddot{\beta}_c - \frac{1}{4}\dot{\beta}_c^2 + K(s)\beta_c^2 = C,\tag{20}$$

with the integral constant C.

The solution (20) determines the coefficients λ_5, λ_6. Next setting $\lambda_2 = \alpha_c(s)$, rewritting λ_4 with $\beta_c(s)$, and substituting (16) into the up-ward equations in (15), we have

$$\lambda_1 = \int^s \alpha_c(s')A(s')ds' + C', \qquad \text{(C':integral constant)} \tag{21}$$

$$\lambda_3 = A(s)\beta_c(s) - \dot{\alpha}_c(s), \tag{22}$$

$$\ddot{\alpha}_c(s) + K(s)\alpha_c(s) = \dot{A}(s)\beta_c(s) + \frac{3}{2}A(s)\dot{\beta}_c(s). \tag{23}$$

Hence these determine the dynamical invariant (14). It is expressed in the form

$$I = \int^s \alpha_c(s')A(s')ds' + C' + \alpha_c P + [A\beta_c - \dot{\alpha}_c]x$$
$$+ \frac{1}{2\beta_c}\left[C\cdot x^2 + \left(\frac{\dot{\beta}_c}{2}x - \beta_c P\right)^2\right].\tag{24}$$

The arbitrariness implied by the presence of the constants C,C' are illusory, as may be verified by making the scale transformation

$$\beta(s) = C^{-\frac{1}{2}}\beta_c,\tag{25-1}$$

$$\alpha(s) = C^{-\frac{1}{2}}\alpha_c,\tag{25-2}$$

$\beta(s)$, $\alpha(s)$ being new auxiliary function of s. The auxiliary equations which $\beta(s)$ and $\alpha(s)$ satisfy are

$$\frac{1}{2}\beta\,\ddot{\beta} - \frac{1}{4}\dot{\beta}^2 + K(s)\beta^2 = 1,\tag{26-1}$$

$$\ddot{\alpha} + K(s)\alpha = \dot{A}(s)\beta + \frac{3}{2}A(s)\dot{\beta}.\tag{26-2}$$

After discarding a constant multiplicative factor $C^{\frac{1}{2}}$ and the constant C', we write Eq. (24) in the form

$$I = \int^{s} \alpha(s') A(s')\, ds' + \alpha(s)\rho + \left[A(s)\beta(s) - \dot{\alpha}(s)\right]x + \frac{1}{2\beta(s)}\left[x^2 + \left(\frac{\dot{\beta}(s)}{2}x - \beta(s)\rho\right)^2\right] \quad (27)$$

3. Example and Discussion

We have shown that the invariant for the forced time-dependent oscillator also exists under the two auxiliary equations, one of which is the differential equation for the ordinary auxiliary condition.

It is the betatron oscillation of an off-momentum particle that is seen as the forced time-dependent harmonic oscillator in accelerators or storage-rings/9/. In this case, $f(s)$ is $\frac{1}{\rho(s)}\frac{\Delta P}{P_0}$ where $\rho(s)$ is the bending radius and ΔP is the deviation from the momentum P_0 of the synchronous particle and the independent variable s is the coordinate along the design orbit. Accelerators and storage rings have necessarily the periodicity corresponding to their circumferences. Therefore A(s) and K(s) also have this periodicity:

$$A(s+L) = A(s) \ ,$$

$$K(s+L) = K(s) \ ,$$

where L is the circumference of a machine. The auxiliary Eqs. (26-1) and (26-2) will possess the particular periodic solutions. In particular, such a solution of Eq. (26-1) is named the betatron amplitude function.

We can give the invariant I the physical meaning, using the concept of the equilibrium orbit (u,v) which is used in accelerator physics. In the present case, the equilibrium orbit (u,v) are the particular solutions of the canonical equations

$$\frac{du}{ds} = \frac{\partial H}{\partial v} = v ,$$ (28-1)

$$\frac{dv}{ds} = -\frac{\partial H}{\partial u} = -K(s)u - A(s) ,$$ (28-2)

where the Hamiltonian is described in the form

$$H(u,v;s) = \frac{1}{2}\left(v^2 + K(s)u^2\right) + A(s)u .$$ (29)

Then, the betatron oscillation about the equilibrium orbit is studied by the linear canonical transformation

$$x = u + X ,$$ (30-1)

$$p = v + P .$$ (30-2)

This transformation is generated by the second type generating function F^2 (x,P;s)

$$F^2(x,P;s) = (v + P)x - u\cdot P .$$ (31)

We have the relevant transformed Hamiltonian $\widehat{H}$, which contains X,P

$$\widehat{H}(X,P;s) = H(u+X, v+P) + \frac{\partial F^2}{\partial s}$$

$$= \frac{1}{2}v^2 - \frac{1}{2}K(s)u^2 + \frac{1}{2}P^2 + \frac{1}{2}K(s)X^2 .$$ (32)

Further retaining only the terms containing X,P , we obtain the

usual un-forced betatron Hamiltonian

$$\widehat{H}(x,P;s) = \frac{1}{2}P^2 + \frac{1}{2}K(s)X^2. \tag{33}$$

This system described in Eq.(33) has the invariant

$$\widehat{I} = \frac{1}{2\beta(s)}\left[X^2 + \left(\frac{\dot\beta(s)}{2}X - \beta(s)P\right)^2\right]. \tag{34}$$

If we rewrite Eq.(34) in the old variables (x,p), we have

$$\widehat{I} = \frac{1}{2\beta}\left[u^2 + \left(\frac{\dot\beta}{2}u - \beta v\right)^2\right] + \frac{1}{2}\left[\dot\beta u - 2\beta v\right]\cdot p$$

$$+\frac{1}{2\beta}\left[-u\left(2+\frac{\dot\beta^2}{2}\right)+\beta\dot\beta v\right]\cdot x + \frac{1}{2\beta}\left[x^2 + \left(\frac{\dot\beta}{2}x - \beta P\right)^2\right]. \tag{35}$$

Setting $\alpha(s) = \frac{1}{2}\left[\dot\beta u - 2\beta v\right]$ (36)

and taking the orbit derivative of $\alpha(s)$, we have the differential equation

$$\ddot\alpha + K(s)\alpha = \beta(s)\dot A(s) + \frac{3}{2}\dot\beta(s)A(s). \tag{37}$$

Eq.(37) exactly agrees with the auxiliary equation (26-2). In addition , substituting Eq.(36) into $A(s)\cdot\beta(s) - \dot\alpha(s)$, we can obtain the form

$$A(s)\beta(s) - \dot\alpha(s) = \frac{1}{2\beta(s)}\left[-u\left(2+\frac{\dot\beta(s)^2}{2}\right) + \beta(s)\dot\beta(s)v\right]. \tag{38}$$

Next if we take the orbit derivative of the first bracket in Eq.(35) , we obtain

$$\frac{dI_u}{ds} = \frac{1}{2}\left[\dot\beta(s)u - 2\beta(s)v\right]\cdot A(s), \tag{39}$$

with $\qquad I_u \equiv \frac{1}{2\beta(s)}\left[u^2 + \left(\frac{\dot\beta(s)}{2}u - \beta(s)v\right)^2\right].$

Futhermore, using the term of $\alpha(s)$, we have

$$\frac{d\,I_u}{ds} = \alpha(s)\,A(s).$$

(40)

Therefore we conclude that the invariant (27) is exactly the Courant and Snyder's invariant of the homogenious betatron motion around the equilibrium orbit (u,v). Also, the auxiliary condition (26-2) is equivalent to the assumption of the particular solution (u,v) of Eqs.(28-1), (28-2).

Acknowledgment

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Formulae

β ------ β (Greek beta)   in Eq.(1),(3),(16),(17),(18),(19)

                                  (20),(22),(23),(24),(25-1)

                                  (26-1),(26-2),(27),(34)∼(39)

h ------h (lower case)   in Eq.(5),(12),(13)

Γ ------ Γ (Greek gamma and cap.)   in Eq.(5),(6),(7),(11),(12)

λ ------ λ (Greek lambda) in Eq.(11),(13),(14),(15)∼(18)

                                  (21),(22)

α ------ α (Greek alpha)   in Eq.(22),(23),(24),(25-2)

                                  (26-2),(27),(36)∼(38),(40)

ρ ------ ρ (Greek rho)   on Section 3. 8 line

Δ ------ Δ (Greek delta and cap.)   on Section 3. 8 line

X ------X (cap.)   in Eq.(30-1)∼(34)

P ------P (cap.)   in Eq.(30-1)∼(34)

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